Water Routing in the Rivers

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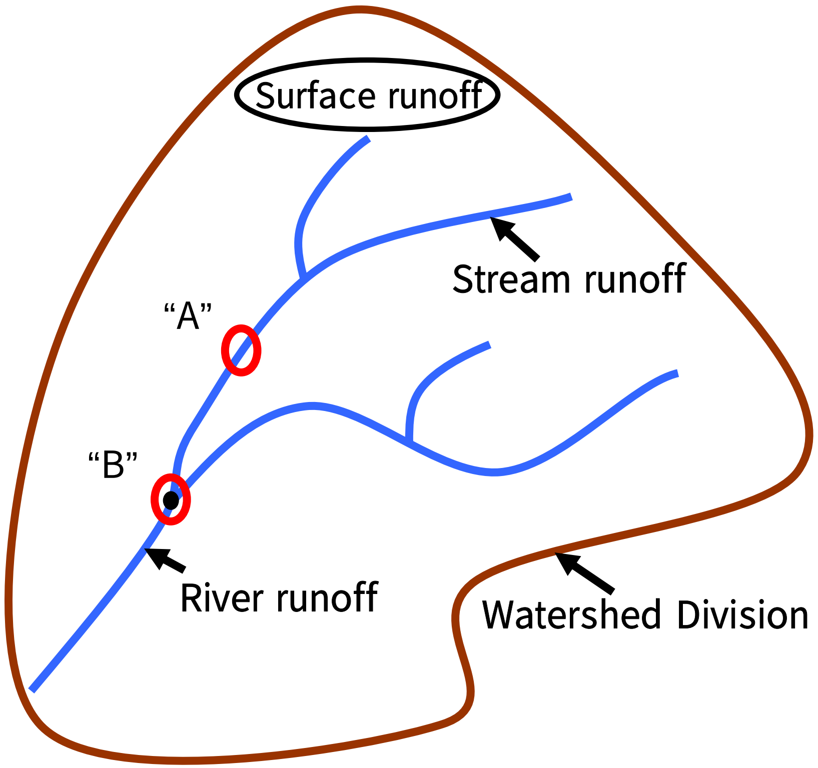
Problem Statement



- **Runoff is coming from precipitation.**

  Precipitation → Surface runoff → Stream (or Drainage) runoff → River runoff → Sea

- **Water Routing is trying to solve discharge, stage (or water depth) through numerical simulation.**

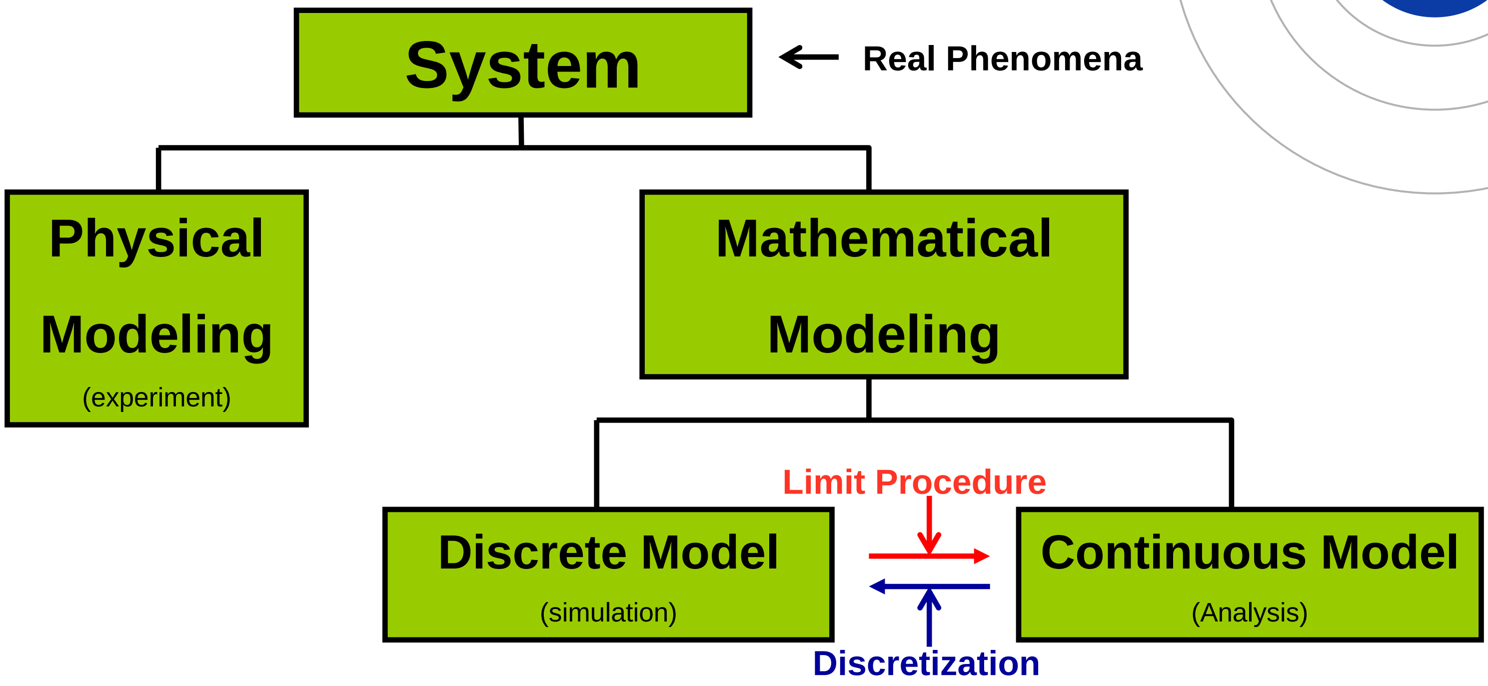
☆ **River runoff in this lecture includes the stream runoff**

Abstract Procedures of Real Phenomena

Real Phenomena —Abstract 1→ Mathematical representation —Abstract 2→ Numerical representation —Abstract 3→ Solution

Step	Abstract 1	Abstract 2	Abstract 3
lost	•Reality •Complexity	•Mathematical reality •Continuity •Nonlinearity(sometimes)	•Numerical reality •exactness
gained	•Simple phenomena •Easy to handle •Some degree of homogenity	•Linearity •Easy to calculate	•Easy to see (simplicity) •Parameterized values

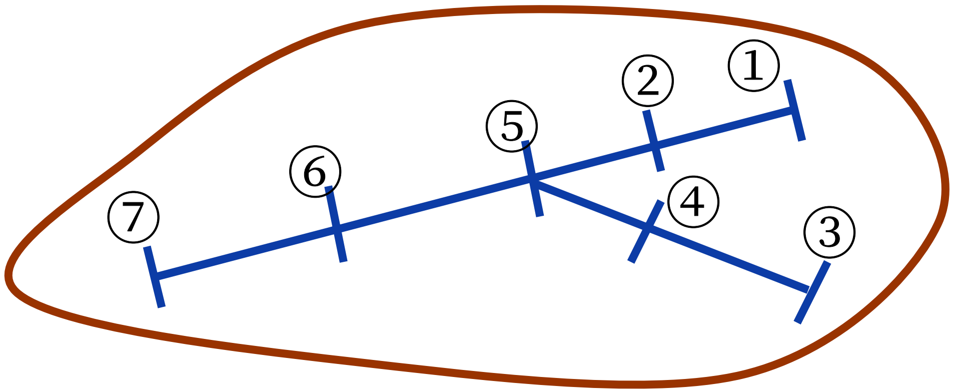
Solution Approaches for Real Phenomena



$$\frac{h_i^{n+1} - h_i^{n}}{\Delta t} \rightleftarrows \frac{\partial h}{\partial t}$$

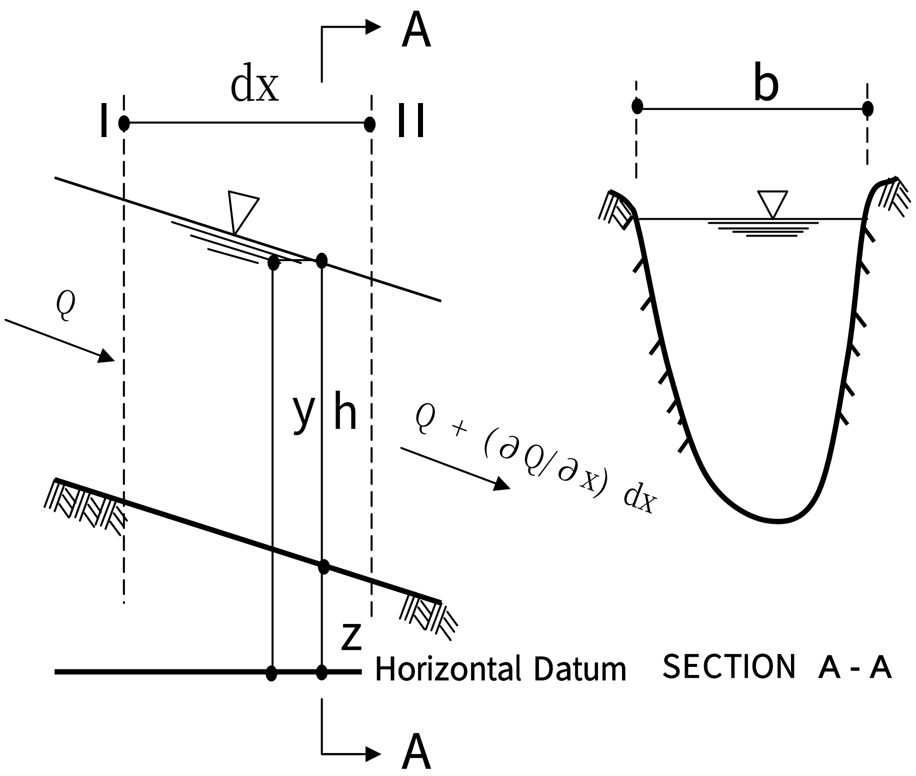
(algebraic equation) (differential equation)

Applying Solution Approaches in River Routing



- Finding the relationship between adjacent two or more points (For example, ① and ②, ③ and ④, ② ④ and ⑤, ⑤ and ⑥ with the governing equations)
  - Continuity equation
  - Momentum or energy equation
- Two different types between the relationship of equation can be used in the river routing
  - In a river
  - At river junction

Governing Equations between Two Points in a River



❑Continuity Equation

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q_l$$

❑Momentum Equation

- **Full dynamic equation**

$$\frac{\partial Q}{\partial t} + \frac{\partial (\frac{Q^2}{A})}{\partial x} + gA\frac{\partial h}{\partial x} = gA(S_0 - S_f) + q_l v_l$$

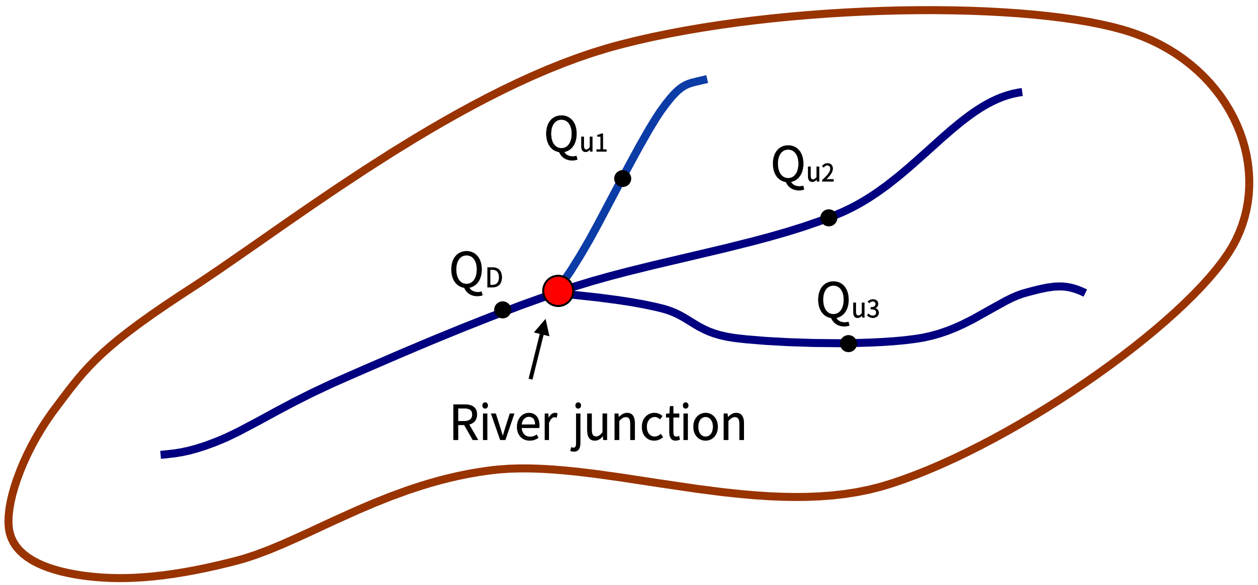
- **Diffusion equation**

$$\frac{\partial h}{\partial x} = (S_o - S_f)$$

- **Kinematic equation**

$$S_o = S_f$$

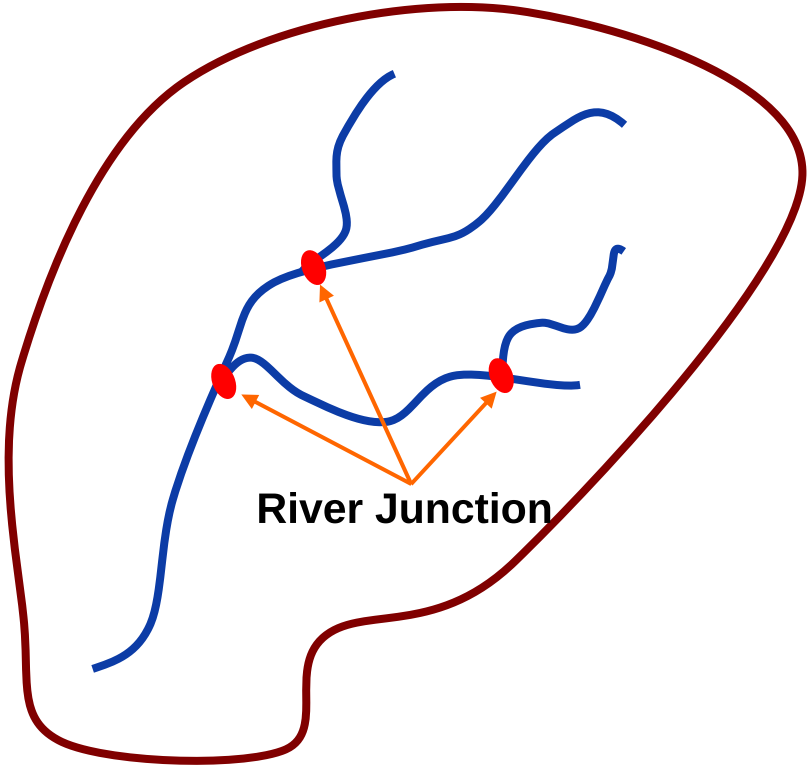
Governing Equations at River Junctions



- **Continuity Equation**
  - $Q_{u1} + Q_{u2} + Q_{u3} = Q_D$
- **Energy Equation**

$$\frac{V_{u1}^2}{2} + gh_{u1} + gZ_{u1} = \left(\int \frac{dV}{dt} dx \right) + \frac{V_D^2}{2} + gh_D + gZ_D + gh_f$$

Characteristics at River Junctions



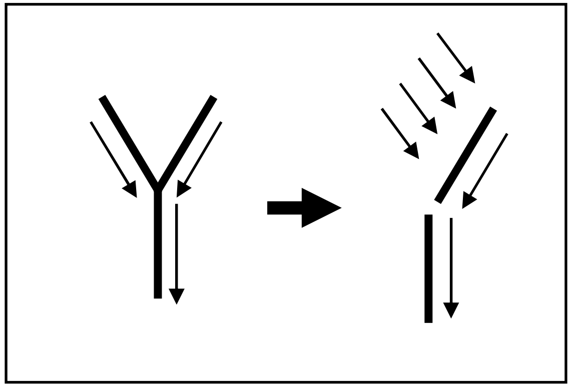
- **Flow Separation**
- **High Energy Losses**
- **Backwater Effects**
- **High Turbulence Effect**
- **Difficulty in Pressure Description** (in Junction Boundaries)
- **Flow Mixing**

River Junction Analysis

- **Experimental Approach**
  - **Mainly, Used in local phenomena analysis**
- **Numerical Approach**
  - **Mainly, Used in entire network analysis**

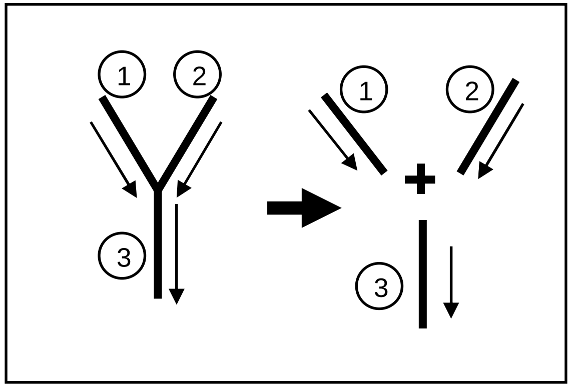
Numerical Approach at River Junctions(1)

❑ Consideration as Lateral Inflow



- Easy to Apply
- Disadvantage in large tributary inflow

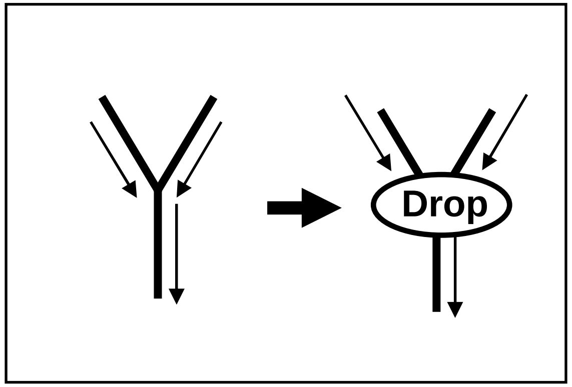
❑ Sequential Type Junction



- Without considering the storage effect
- Cannot consider backwater effect

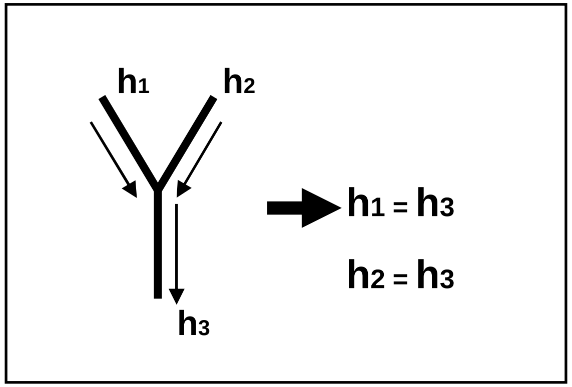
Numerical Approach at River Junctions(2)

❑ Drop Type Junction



- Critical flow assumption
- Cannot consider backwater effect

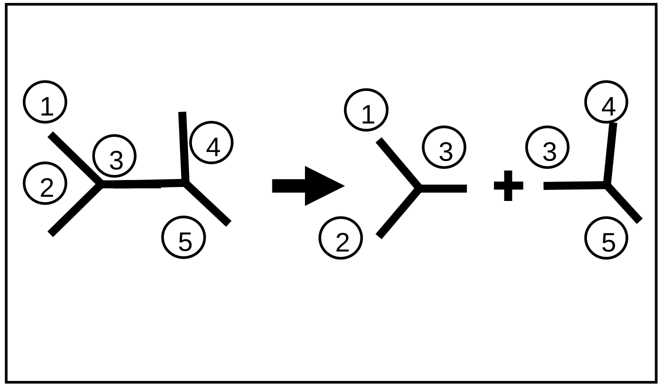
❑ Point Type Junction



- Backwater effect is partially accounted for.
- Storage effect is neglected.

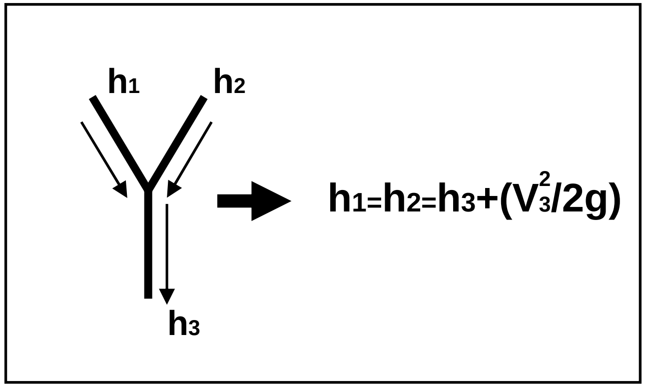
Numerical Approach at River Junctions(3)

❑ Y Segment Assumption



- Partial backwater and storage effect are considered
- Easy for programming

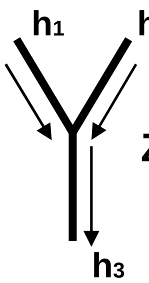
❑ Reservoir Type Junction



- Backwater and storage effects are considered
- Energy losses at junctions are neglected

Numerical Approach at River Junctions(4)

Full Energy Approach



$$Z_1+h_1+\frac{V_1^2}{2g}=Z_3+h_3+\frac{V_3^2}{2g}+h_f$$

- Commonly used approach in the simultaneous solution algorithm.

Review of Numerical Approaches

- Lateral inflow approach and sequential type junction are not realistic even though having the advantage of simlicity.
- Most approaches are based on the energy assumption even though the momentum approach is used between the adjacent river sections.
- The main reason for using the energy assumption is the difficulty in pressure description in junction boundaries.

Two Different Knowledges for Solution

Discretization

- Discretization is to be changed as the discrete values (unknowns) from the continuous differential equations.
- Methods
  - Characteristics Approach
  - Finite Difference Approach

    …
  - Finite Element Approach

Solution Algorithm

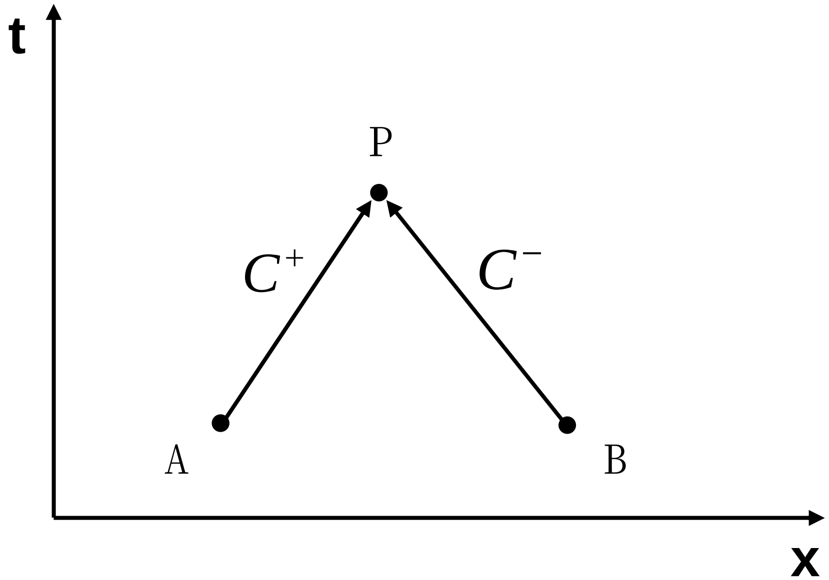
- Solution algorithms are known as the available methods to be used for finding the values of unknowns.
- Methods
  - Gauss elimination

    …
  - Matrix inversion method
  - Gauss Jordan method

Discretization(1)

Characteristics Method

- The method to find the values of unknown in a combining point using two different wave velocity equations (C^+, C^-).



Discretization(2)

Finite Difference Method(1)

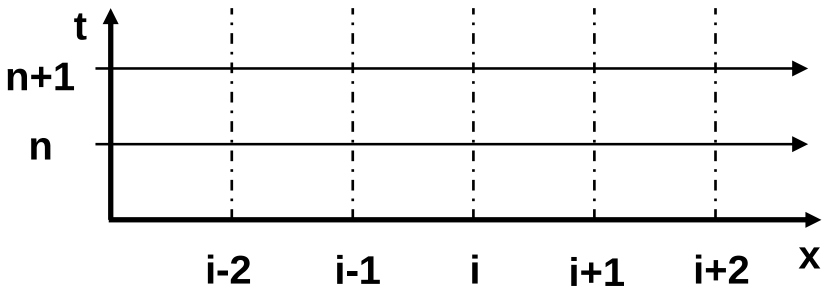
- **The method to find the values of unknown using discretization of the differential equations.**

- **For Explicit Scheme**
  - **Unknown variables at any point at the time level n+1 is computed based entirely on known data at a few adjacent points at time level n or more previous time level n-1.**
  - **Lax Scheme**
  - **Leap-Frog Scheme**
  - **Diffusion Scheme**
  - **Staggered Scheme**

Discretization(3)

Finite Difference Method(2)

- **Implicit Scheme**
  - Unknown variables at any point at the time level n+1 are solved in a group of advanced points through the use of simultaneous equations.
  - 4 point scheme
  - Abbott-Ionescu scheme
  - Vasiliev scheme
- **Abbott-Ionescu Scheme**



$$\frac{\partial u}{\partial t} = \frac{1}{2}\left(\frac{u_{i+1}^{n+1} - u_{i+1}^{n}}{\Delta t} + \frac{u_{i-1}^{n+1} - u_{i-1}^{n}}{\Delta t}\right)$$

$$\frac{\partial h}{\partial t} = \frac{h_i^{n+1} - h_i^{n}}{\Delta t}$$

Solution Algorithms(1)

- **Simultaneous algebraic equation can be written as the following forms.**

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots\cdots a_{1n}x_n &= C_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots\cdots a_{2n}x_n &= C_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots\cdots\cdots a_{3n}x_n &= C_3 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots\cdots\cdots a_{nn}x_n &= C_n
\end{aligned}
$$

- **The algebraic equations can be rewritten as a matrix form [A]{x} = {C}**

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{33} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{bmatrix}
\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_n \end{Bmatrix}
\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ \cdots \\ C_4 \end{Bmatrix}
$$

coefficient matrix　　unknowns　　constants

Solution Algorithms(2)

- **Two different approaches in the solution algorithms are used.**
- **Direct method**
  - **The method is frequently called as elimination method or reduction method.**
  - **The solution is directly obtained through a simulation procedure.**
    - **Gauss elimination method**
    - **Gauss-Jordan elimination method**
    - **Use of matrix inversion**
- **Indirect method**
  - **The method is frequently called as the iterative method.**
  - **The solution is obtained through iterations using the assumed unknown values.**
    - **Gauss-Seidal method**
    - **Jacobian method**

Solution Algorithms(3)

Gauss Elimination Method(1)

- **Original relation**

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots\cdots\cdots a_{1n}x_n = C_1$$

- **Original relation is divided by the coefficient of unknown value(x_1)**

$$x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3 + \ldots\ldots\ldots \frac{a_{1n}}{a_{11}}x_n = \frac{C_1}{a_{11}}$$

- **Unknowns($x_1, x_2 \ldots x_{n-1}$) except the last unknown value (x_n) are eliminated by multiplying by the coefficient of unknown values.**

Solution Algorithms(4)

Gauss Elimination Method(2)

- **After the triangular set of equations has been obtained, the last unknown value is obtained directly.**

$$a_{nn}^{i-1} x_n = c_n^{i-1}$$

$$x_n = \frac{c_n^{i-1}}{a_{nn}^{i-1}}$$

→ i **indicated the number of reduction procedure**

- **The remained unknowns(x_1,x_2…x_{n-1}) are obtained in the equations with the substitution from the last unknown(x_n).**

Solution Algorithms(5)

Gauss-Seidal Method

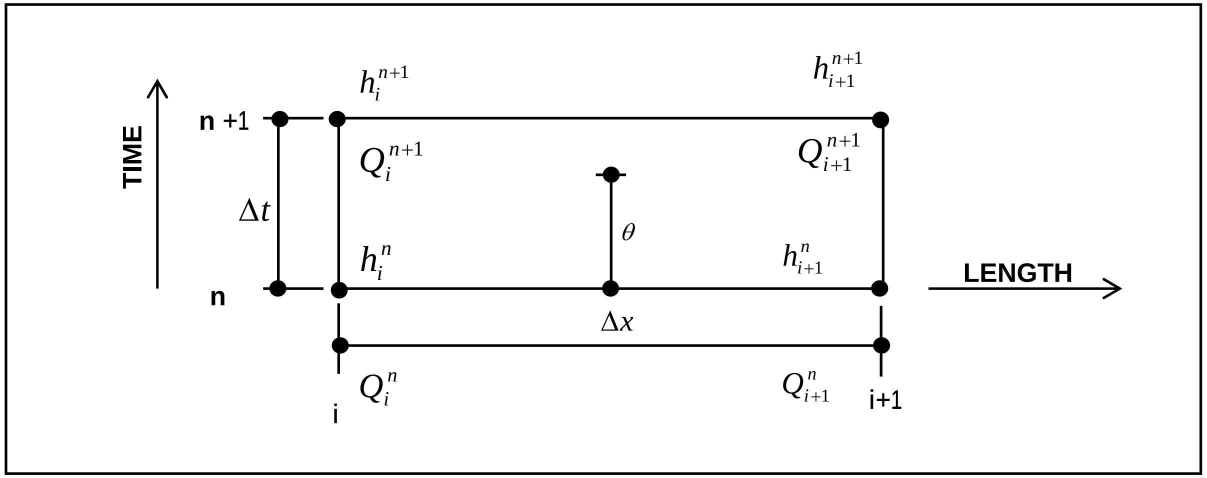
- **Original relation**

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots\cdots\cdots a_{1n}x_n = C_1$$

- **Assign the initial values for the unknown values appearing in the original relation.**
- **Starting with the first equation, solve the first unknown value using the assumed values for the other unknowns.**
- **Proceed with the remaining equations, and then finally the remained unknown values are obtained.**
- **Because of the error in the assumption of initial values, continue the iterations until the values of each unknown are with in the range of error.**

Finite Difference Approximation

❑ Four Points Scheme



- **time derivative :**

$$\frac{\partial h}{\partial t}=\frac{1}{2}\left(\frac{h_{i+1}^{n+1}-h_{i+1}^{n}}{\Delta t}+\frac{h_{i}^{n+1}-h_{i}^{n}}{\Delta t}\right)$$

- **space derivative:**

$$\frac{\partial Q}{\partial x}=\theta\left(\frac{Q_{i+1}^{n+1}-Q_{i}^{n+1}}{\Delta x}\right)+(1-\theta)+\left(\frac{Q_{i+1}^{n}-Q_{i}^{n}}{\Delta x}\right)$$

- **coefficients:**

$$f_{i+1/2}^{n}=\frac{1}{2}\left(f_i^n+f_{i+1}^n\right)$$

Finite Difference Formulation of Continuity Equation

$$\theta\frac{Q_{i+1}^{n+1}-Q_i^{n+1}}{\Delta x}+(1-\theta)\frac{Q_{i+1}^{n}-Q_i^{n}}{\Delta x}+\frac{1}{2\Delta t}b_{i+1/2}^{n}$$

$$(h_i^{n+1}-h_i^{n}+h_{i+1}^{n+1}-h_{i+1}^{n})=\theta q_{l_{i+1/2}}^{n+1}+(1-\theta)q_{l_{i+1/2}}^{n+1}$$

$$D_1Q_i^{n+1}+E_1h_i^{n+1}+F_1Q_{i+1}^{n+1}+G_1h_{i+1}^{n+1}=T_1$$

where

$$D_1=-4\theta\psi$$

$$E_1=2b_{i+1/2}^{n}$$

$$F_1=4\theta\psi$$

$$G_1=2b_{i+1/2}^{n}$$

$$T_1=2b_{i+1/2}^{n}h_i^{n}+2b_{i+1/2}^{n}h_{i+1}^{n}+4(1-\theta)\psi Q_i^{n}-4(1-\theta)\psi Q_{i+1}^{n}+4\theta\Delta t q_{l_{i+1/2}}^{n+1}-4(1-\theta)\Delta t q_{l_{i+1/2}}^{n}$$

$$D_2Q_i^{n+1}+E_2h_i^{n+1}+F_2Q_{i+1}^{n+1}+G_2h_{i+1}^{n+1}=T_2$$

Continuity Equation at River Junctions

$$D_8 Q_{u1}^{n+1} + E_8 Q_{u2}^{n+1} + F_{81} Q_{u3}^{n+1} + G_8 Q_D^{n+1} = T_8$$

$$D_8 = \theta$$

$$E_8 = \theta$$

$$F_8 = \theta$$

$$G_8 = \theta$$

$$T_8 = -(1-\theta)(Q_{u1}^n + Q_{u2}^n + Q_{u3}^n + Q_D^n)$$

Momentum Equation at River Junctions

$$D_{10}Q_{u1}^{n+1} + E_{10}h_{u1}^{n+1} + F_{10}Q_{D}^{n+1} + G_{10}h_{D}^{n+1} = T_{10}$$

$$D_{10} = \cos y_{u1} + \frac{2g\theta A_{u1}^{n} S_{f\,u1}^{\;n}}{Q_{u1}^{n}} - 2\theta\psi V_{u1}^{n} \cos y_{u1}$$

$$E_{10} = -2g\theta\psi A_{u1}^{n} - \frac{2g\theta\Delta t A_{u1}^{n} S_{f\,u1}^{\;n}}{K_{u1}^{n}} \left(\frac{\partial K_{u1}^{n}}{\partial h} \right)$$

$$F_{10} = 1 + 2\theta\psi V_{D}^{n} + \frac{2g\theta\Delta t \left(A_{u1}^{n} + A_{u2}^{n} + A_{u3}^{n} \right) S_{f\,D}^{\;n}}{Q_{D}^{n}}$$

$$G_{10} = 2g\theta\psi \left(A_{u1}^{n} + A_{u2}^{n} A_{u3}^{n} \right) - \frac{2g\theta\Delta t \left(A_{u1}^{n} + A_{u2}^{n} + A_{u3}^{n} \right) S_{f\,d1}^{\;n}}{K_{D}^{n}} \left(\frac{\partial K_{D}^{n}}{\partial h} \right)$$

Matrix Coefficients in a river

$$
\begin{bmatrix}
a_{11} & a_{12} & & & & & & & \\
& & a_{21} & a_{22} & a_{23} & a_{24} & & & \\
& & a_{31} & a_{32} & a_{33} & a_{34} & & & \\
& & & & a_{41} & a_{42} & a_{43} & a_{44} & \\
& & & & a_{51} & a_{52} & a_{53} & a_{54} & \\
& & & & & \vdots & & & \\
& & & & & a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & a_{n-1,4} \\
& & & & & & & a_{n,1} & a_{n,2}
\end{bmatrix}
$$

Reduced Matrix Coefficients in a river

$$
\begin{bmatrix}
 & & a_{11} & a_{12} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44} \\
a_{51} & a_{52} & a_{53} & a_{54} \\
 & & \vdots & \\
a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & a_{n-1,4} \\
 & & a_{n,1} & a_{n,2}
\end{bmatrix}
$$

Proposed Solution Algorithm in a River(1)

$$
\begin{pmatrix}
a_{1,1} & a_{1,2} & - & - & - & - & - & - & - & - & - & - & - & - \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & - & - & - & - & - & - & - & - & - & - \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & - & - & - & - & - & - & - & - & - & - \\
- & - & a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & - & - & - & - & - & - & - & - \\
- & - & a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & - & - & - & - & - & - & - & - \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
- & - & - & - & a_{7,1} & a_{7,2} & - & - & - & - & - & a_{7,3} & a_{7,4} & - \\
- & - & - & - & - & - & a_{8,1} & a_{8,2} & - & - & - & - & - & - \\
- & - & - & - & - & - & a_{9,1} & a_{9.2} & a_{9,3} & a_{9,4} & - & - & - & - \\
- & - & - & - & - & - & a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & a_{12,1} & - & a_{12,2} & - & - & - & a_{12,3} & - & - & a_{12,4} & - & - \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
- & - & - & - & - & - & - & - & - & - & - & - & a_{2n,1} & a_{2n,2}
\end{pmatrix}
$$

- Recursion equations:

$$X_i = \frac{Z_i - M_i - M_{i,4} X_{i+1}}{M_{i,3}}; i = 1, 3, 5, ..., 2n-1$$

$$X_i = \frac{Z_i - a_{i,4} X_{i+2} - a_{i,3} - a_{i,3} X_{i+1}}{M_{i,2}}; i = 2, 4, ..., 2n$$

$$X_{2n} = \frac{Z_{2n}}{M_{2n,2}}$$

Proposed Solution Algorithm in a River(2)

- **The recurrent coefficients of the momentum equations in a river are:**

$$M_{i,2} = -a_{i,1}\frac{M_{i-1,4}}{M_{i-1,3}} + a_{i,2}$$

$$Z_i = -a_{i,1}\frac{Z_{i-1}}{M_{i-1,3}} + \tau_i$$

In which the index $i = (2 \times \text{station number}) - 1$

Proposed Solution Algorithm in a River(3)

- **The recurrent coefficients of the continuity equation in a river are:**

$$M_{i,2} = -a_{i,1}\frac{M_{i-2,4}}{M_{i-2,3}} + a_{i,2}$$

$$M_{i,3} = -a_{i-1,3}\frac{M_{i,2}}{M_{i-1,2}} + a_{i,3}$$

$$M_{i,4} = -a_{i-1,4}\frac{M_{i,2}}{M_{i-1,2}} + a_{i,4}$$

$$Z_i = -M_{i,2}\frac{Z_{i-1}}{M_{i-1,2}}a_{i,1}\frac{M_{i-2}}{M_{i-2,3}} + \tau_i$$

In which the index $i = 2 \times \text{station number.}$

Proposed Solution Algorithm at River Junctions(1)

- **The recurrent coefficient of the energy or momentum equation at channel junctions are:**

$$M_{i,2} = -a_{i,1}\frac{M_{i-1,4}}{M_{i-1,3}} + a_{i,2}$$

$$Z_i = -a_{i,1}\frac{Z_{i-1}}{M_{i-1,3}} + \tau_i$$

In which the index $i = 2 \times \text{station number.}$

Proposed Solution Algorithm at River Junctions(2)

- **The recurrent coefficient of the continuity equation at river junctions are:**

$$B_{2u_k} = -M_{2u_k,2}\frac{M_{2u-1_k-1,3}}{M_{2u_k-1,4}}$$

$$D_{2u_k} = Z_{2u_k} - M_{2u_k,2}\frac{Z_{2u_k-1}}{M_{2u_k-1,4}}$$

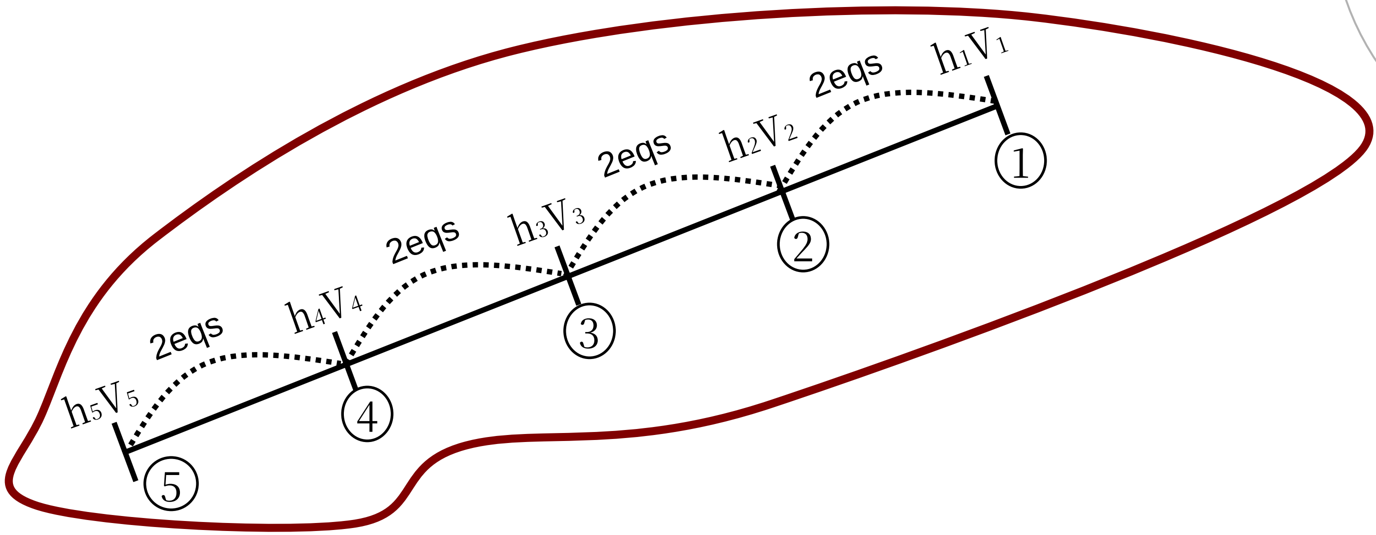
$$M_{i,3} = a_{i,4} - \sum_{k=1}^{NC} a_{i,k}\frac{a_{2u_k,3}}{B_{2u_k}}$$

$$M_{i,4} = -\sum_{k=1}^{NC} a_{i,k}\frac{a_{2u_k,4}}{B_{2u_k}}$$

$$Z_i = \tau_i - \sum_{k=1}^{NC} a_{i,k}\frac{D_{2u_k}}{B_{2u_k}}$$

In which *NC* is number of upstream branches; U_k is upstream section number along branch number *k*, and index $i = (2 \times \text{station number}) + 1$

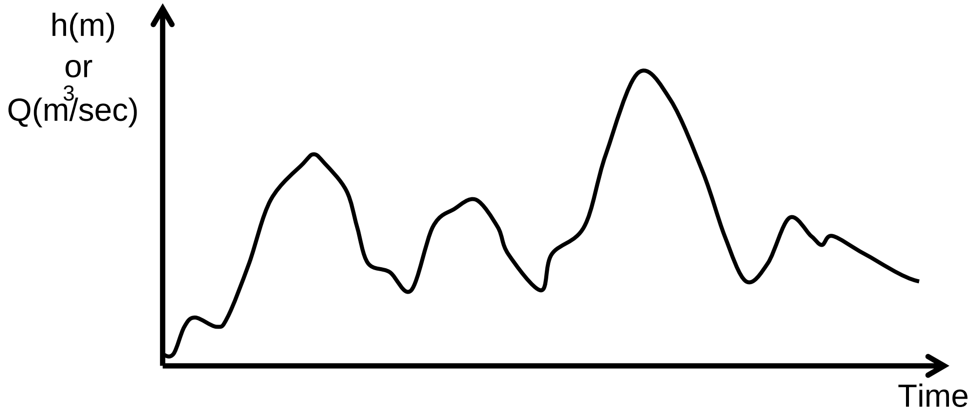
Application to the Model(1)

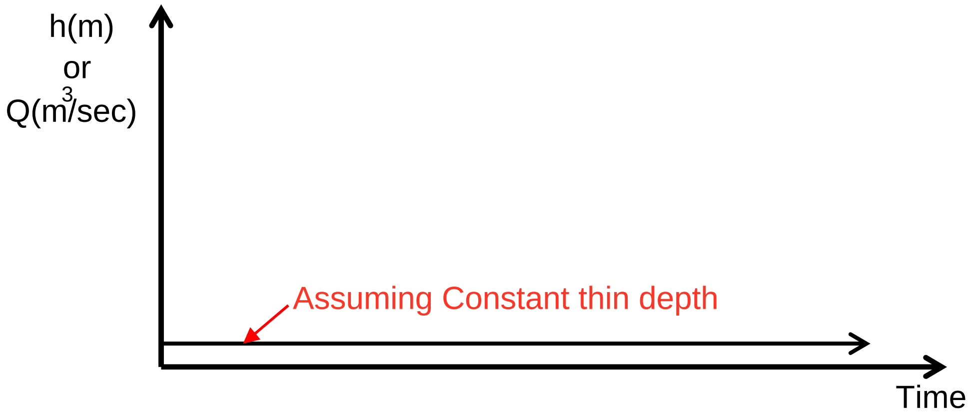


- In the example, 8 equations $(4 \times 2eqs = 8)$ exist and 10 unknowns $(4 \times 2eqs = 8)$ need to be solved using the model.
- Therefore, two equations lack to find the solutions. Usually, two boundary conditions are used to solve the unknown values.
  - Stage (or water depth) variation depending upon time
  - Stage-discharge relationship

Application to the Model(2)

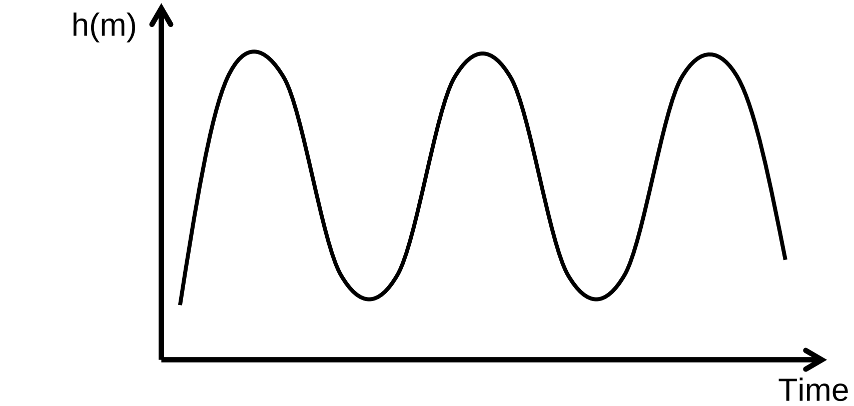
❑ Upstream Boundary Condition

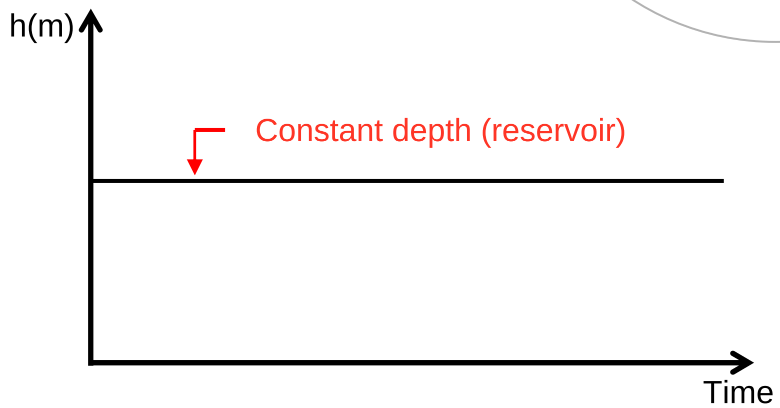


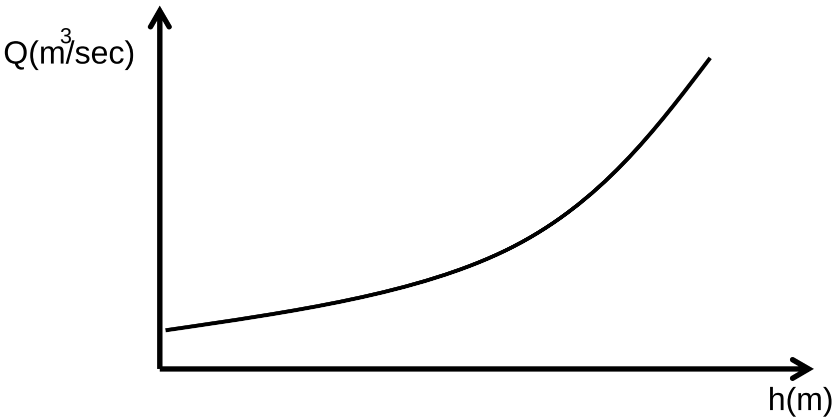


Application to the Model(3)

❑ Downstream Boundary Condition







$$v = \frac{1}{n} R^{\frac{2}{3}} i_0^{\frac{1}{2}}$$

< uniform flow assumption>